# Generalized principal bundles and quotient stacks

Elena Caviglia

University of Leicester

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# Classical principal bundles

**Definition.**

Let $X$ be a topological space and let $G$ be a topological group. A **principal** $G$**-bundle** over $X$ is a topological space $P$ equipped with an action $p\colon G \times P \to P$ and a $G$-equivariant continuous map $\pi_P\colon P \to X$, that is locally trivial, i.e. there exists an open covering $\{U_i\}_{i \in I}$ of $X$ such that for every $i \in I$ the restriction $P\big|_{U_i}$ is isomorphic to $G \times U_i$ via a $G$-equivariant isomorphism.

# Setting and preliminary definitions

Let $\mathcal{C}$ be a category with pullbacks and terminal object $T$ and let $\tau$ be a Grothendieck topology on it.

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**Definition.**

Let $G$ be an internal group in $\mathcal{C}$ and let $X$ be an object of $\mathcal{C}$. An **action of $G$ on $X$** is a morphism $x\colon G \times X \to X$ such that the following diagrams are commutative

$$
\begin{array}{ccc}
G \times G \times X & \xrightarrow{\mathrm{id}_G \times x} & G \times X \\
{\scriptstyle m\times\mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle x} \\
G \times X & \xrightarrow[x]{} & X
\end{array}
\qquad
\begin{array}{ccc}
T \times X & \xrightarrow{e\times\mathrm{id}_X} & G \times X \\
{\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle x} \\
X & \xrightarrow[\mathrm{id}_X]{} & X,
\end{array}
$$

where $m\colon G \times G \to G$ is the internal multiplication of $G$ and $e : T \to G$ is the internal neutral element of $G$.

# Setting and preliminary definitions

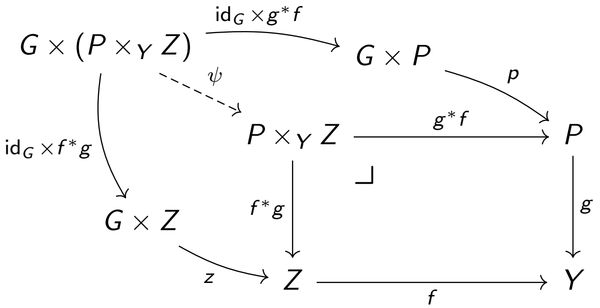
**Definition.**

Let $G$ be a group object in $\mathcal{C}$ that acts on the objects $X$ and $Y$ of $\mathcal{C}$ with actions $x\colon G \times X \to X$ and $y\colon G \times Y \to Y$ respectively. A $G$-**equivariant morphism** $f\colon X \to Y$ is a morphism in $\mathcal{C}$ such that the following square is commutative:

$$\begin{array}{ccc} G \times X & \xrightarrow{\quad x \quad} & X \\ {\scriptstyle \mathrm{id}_G \times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times Y & \xrightarrow[\quad y \quad]{} & Y. \end{array}$$

# Actions on pullbacks

Let $G$ be a group object of $\mathcal{C}$ that acts on $P, Y, Z \in \mathcal{C}$ and let $f\colon Z \to Y$ and $g : P \to Y$ be $G$-equivariant morphisms.
We define an action of $G$ on the pullback $P \times_Y Z$ using the morphism $\psi\colon G \times (P \times_Y Z) \to P \times_Y Z$ induced by the universal property of the pullback $P \times_Y Z$ as in the following diagram



# Locally trivial morphisms

**Definition (C.).**

Let $g\colon Y \to X$ be a morphism of $\mathcal{C}$. We say that $g$ is **locally trivial** if there exists a covering $\{f_i\colon U_i \to X\}_{i\in I}$ of $X$ such that for every $i \in I$ the pullback

$$\begin{array}{ccc} Y \times_X U_i & \xrightarrow{g^* f_i} & Y \\ {\scriptstyle f_i^* g}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ U_i & \xrightarrow[f_i]{} & X \end{array}$$

is isomorphic to $G \times U_i$ via a $G$-equivariant isomorphism.

# Principal $G$-bundles

## Definition (C.).

Let $G$ be an internal group in the site $(\mathcal{C}, \tau)$ and let $X$ be an object of $\mathcal{C}$. A **principal $G$-bundle over $X$** is an object $P \in \mathcal{C}$ equipped with an action $p\colon G \times P \to P$ and a $G$-equivariant locally trivial morphism $\pi_P\colon P \to X$.

Let $\pi_P\colon P \to X$ and $\pi_Q\colon Q \to X$ be principal $G$-bundles over $X$ in $\mathcal{C}$. A **morphism of principal $G$-bundles over $X$** from $\pi_P\colon P \to X$ to $\pi_Q\colon Q \to X$ is a $G$-equivariant morphism $\varphi\colon P \to Q$ in $\mathcal{C}$ such that

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ & {\scriptstyle \pi_P} \searrow \quad \swarrow {\scriptstyle \pi_Q} & \\ & X. & \end{array}$$

# Closure under pullbacks

**Proposition (C.).**

*Let $\pi_P \colon P \to Y$ be a principal $G$-bundle over $Y$ and let $f \colon Z \to Y$ be a $G$-equivariant morphism in $\mathcal{C}$. Then the morphism $f^*\pi_P \colon P \times_Y Z \to Z$ is a principal $G$-bundle over $Z$.*

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The morphism $f^*\pi_P$ is $G$-equivariant since the square

$$\begin{array}{ccc} G \times (P \times_Y Z) & \xrightarrow{\psi} & P \times_Y Z \\ \downarrow{\scriptstyle \mathrm{id}_G \times f^*\pi_P} & & \downarrow{\scriptstyle f^*\pi_P} \\ G \times P & \xrightarrow[p]{} & P, \end{array}$$

is commutative by definition of $\psi$.

# Closure under pullbacks

We consider the covering $\mathcal{V} = \{g_i \colon V_i \to Y\}_{i \in I}$ such that for every $i \in I$ the pullback $P \times_Y V_i$ is isomorphic to $G \times V_i$ via the $G$-equivariant isomorphism $\alpha_i$ and we use it to define the covering family

$$\mathcal{U} = \{f^* g_i \colon V_i \times_Y Z \to Z\}_{i \in I}.$$

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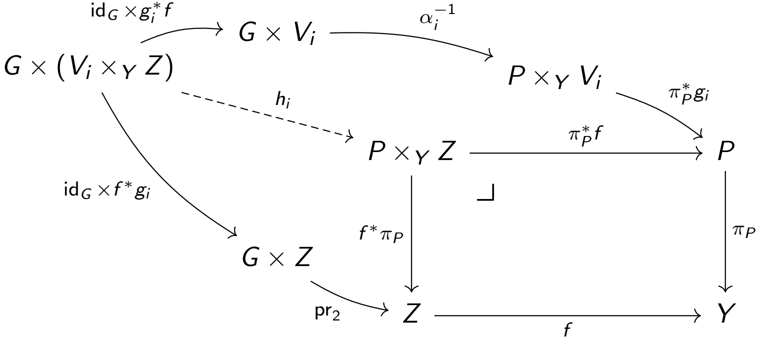
We show that $G \times (V_i \times_Y Z)$ satisfies the universal property of the pullback $(P \times_Y Z) \times_Z (V_i \times_Y Z)$.

$$\begin{array}{ccc} G \times (V_i \times_Y Z) & \xrightarrow{h_i} & P \times_Y Z \\ \Big\downarrow \mathrm{pr}_2 & & \Big\downarrow f^*\pi_P \\ V_i \times_Y Z & \xrightarrow[f^* g_i]{} & Z \end{array}$$

# Closure under pullbacks

For every $i \in I$, we define the morphism $h_i$ as follows



# Generalized quotient prestacks

## Definition (C.).

Let $X$ be an object of $\mathcal{C}$ and let $G$ be a group object of $\mathcal{C}$ that acts on $X$ with action $x\colon G \times X \to X$.

The **quotient prestack** $[X/G]\colon \mathcal{C}^{\text{op}} \to \mathcal{C}at$ is defined as follows:

- for every object $Y \in \mathcal{C}$ we define $[X/G](Y)$ as the category that has
  - as objects the pairs $(P, \alpha)$ where $\pi_P\colon P \to Y$ is a principal $G$-bundle over $Y$ and $\alpha\colon P \to X$ is a $G$-equivariant morphism;
  - as morphisms from $(P, \alpha)$ to $(Q, \beta)$ the morphisms of principal $G$-bundles $\varphi\colon P \to Q$ such that

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & X & \end{array}$$

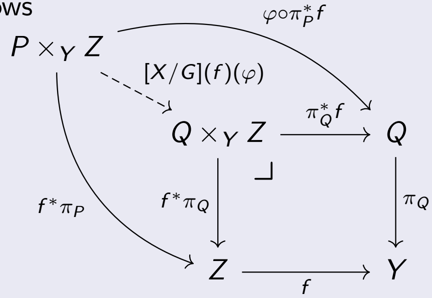
## Definition (C.).

- for every morphism $f\colon Z \to Y$ in $\mathcal{C}$, we define

$$[X/G](f)\colon [X/G](Y) \to [X/G](Z)$$

as the functor that sends

- an object $(P, \alpha) \in [X/G](Y)$ to the pair $(P \times_Y Z, \alpha \circ \pi_P^* f)$, where $P \times_Y Z$ is the pullback of $f$ and $\pi_P$;
- a morphism $\varphi\colon (P, \alpha) \to (Q, \beta)$ to the morphism $[X/G](f)(\varphi)$ defined as follows



# Well-definedness of $[X/G]$

**Proposition (C.).**

*The quotient prestack $[X/G]\colon \mathcal{C}^{\text{op}} \to \mathcal{C}at$ is well-defined.*

Given $f\colon Z \to Y$ the assignment

$$[X/G](f)\colon [X/G](Y) \to [X/G](Z)$$

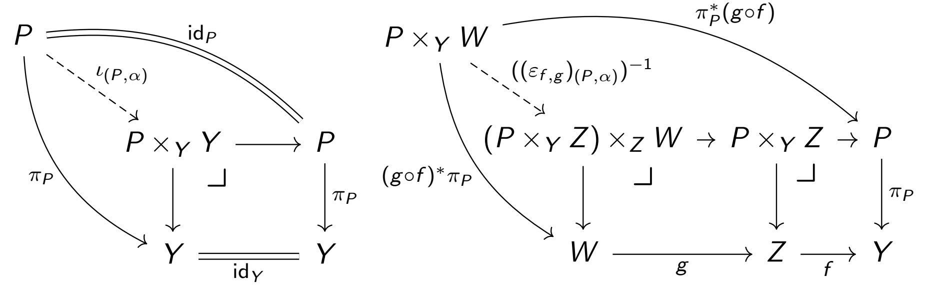
- is well-defined on objects because principal $G$-bundles over $Y$ are closed under pullbacks;
- preserves identities and compositions by the universal property of the pullback.

# Pseudofunctoriality of [X/G]

**Proposition (C.).**

*The quotient prestack $[X/G]\colon \mathcal{C}^{\text{op}} \to \mathcal{C}at$ is a prestack.*

$$\iota_Y\colon \mathrm{id}_{[X/G](Y)} \Rightarrow [X/G](\mathrm{id}_Y) \qquad \varepsilon_{f,g}\colon [X/G](g)\circ[X/G](f) \Rightarrow [X/G](f\circ g)$$



# Some observations

**Remark.**

The quotient prestack $[X/G]$ doesn't necessarily take values in $\mathcal{G}pd$.

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**Definition (C.).**

Let $\mathcal{C}$ be a category with terminal object $T$ and let $G$ be a group object in $\mathcal{C}$. We call **classifying prestack** the prestack $[T/G]$ and we denote it $\mathcal{B}G$.

# Descent data

**Definition.**

Let $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ be a prestack and let $S$ be a sieve on $Y \in \mathcal{C}$.

A **descent datum on $S$ for $\mathcal{F}$** is an assignment for every morphism $Z \xrightarrow{f} Y$ in $S$ of an object $W_f \in \mathcal{F}(Z)$ and, for every pair of composable morphisms $Z' \xrightarrow{g} Z \xrightarrow{f} Y$ with $f \in S$, of an isomorphism $\varphi^{f,g}\colon g^* W_f \xrightarrow{\simeq} W_{g\circ f}$ such that, given morphisms $Z'' \xrightarrow{h} Z' \xrightarrow{g} Z \xrightarrow{f} Y$ with $f \in S$, the following diagram is commutative

$$\begin{array}{ccc} h^*(g^* W_f) & \xrightarrow{h^*\varphi^{f,g}} & h^*(W_{f\circ g}) \\ \big\downarrow \wr & & \big\downarrow \varphi^{f\circ g,h} \\ (g\circ h)^*(W_f) & \xrightarrow[\varphi^{f,g\circ h}]{} & W_{f\circ g\circ h}. \end{array}$$

# Descent data

## Definition.

This descent datum is called **effective** if there exist an object $W \in \mathcal{F}(Y)$ and, for every morphism $Z \xrightarrow{f} Y \in S$, an isomorphism

$$\psi^f : f^*(W) \xrightarrow{\simeq} W_f$$

such that, given morphisms $Z' \xrightarrow{g} Z \xrightarrow{f} Y$ with $f \in S$, the following diagram is commutative

$$\begin{array}{ccc} g^*(f^*(W)) & \xrightarrow{g^*\psi^f} & g^*(W_f) \\ \downarrow \wr & & \downarrow \varphi^{f,g} \\ (f \circ g)^* W & \xrightarrow{\psi^{f \circ g}} & W_{f \circ g}. \end{array}$$

# Definition of stack

**Definition.**

A prestack $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ is a **stack** if it satisfies the following conditions:

- Every descent datum for $\mathcal{F}$ is effective;
- (*Gluing of morphisms*) Given a covering family $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I}$, objects $x$ and $y$ of $\mathcal{F}(U)$ and morphisms $\varphi_i\colon x\big|_{U_i} \to y\big|_{U_i}$ in $\mathcal{F}(U_i)$ for every $i \in I$ such that for every $i, j \in I$ $\varphi_i\big|_{U_{ij}} = \varphi_j\big|_{U_{ij}}$, there exists a morphism $\eta\colon x \to y$ such that $\eta\big|_{U_i} = \varphi_i$;
- (*Uniqueness of gluings*) Given a covering family $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I}$, objects $x$ and $y$ of $\mathcal{F}(U)$ and morphisms $\varphi, \psi : x \to y$ such that for every $i \in I$ $\varphi\big|_{U_i} = \psi\big|_{U_i}$, then $\varphi = \psi$.

# Canonical topology on a site

**Definition.**

Let $\mathcal{C}$ be a category. The **canonical topology** $\kappa$ on the category $\mathcal{C}$ is the finest Grothendieck topology on $\mathcal{C}$ such that every representable presheaf of $\mathcal{C}$ is a sheaf on the site $(\mathcal{C}, \kappa)$.

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**Definition.**

Let $\mathcal{C}$ be a category and let $X$ be an object of $\mathcal{C}$. A sieve $S$ on $X$ is a **colim sieve** if $X = \operatorname{colim}_S \operatorname{dom}$, where $\operatorname{dom}\colon S \to \mathcal{C}$ is the domain functor. Moreover, $S$ is a **universal colim sieve** if for every morphism $f\colon Y \to X$ in $\mathcal{C}$ the sieve $f^*S$ on $Y$ is a colim sieve.

**Proposition (Johnstone, Lester).**

*The sieves for the canonical topology are exactly the universal colim sieves.*

# Basis for the canonical topology

**Definition.**

A morphism $f\colon C \to D$ in $\mathcal{C}$ is called an **effective epimorphism** if it is the coequalizer of its kernel pair. Moreover, it is called a **universal effective epimorphism** if its base change along any morphism is an effective epimorphism.

**Theorem (Lester).**

*Let $\mathcal{C}$ be a cocomplete category with pullbacks and a terminal object and such that pullbacks preserve colimits. Then the family $\{f_i\colon Y_i \to X\}_{i\in I}$ is a covering family for $X$ in $\kappa$ if and only if the morphism*

$$\coprod_{i\in I} f_i\colon \coprod_{i\in I} Y_i \to X$$

*is a universal effective epimorphism.*

# The main theorem

**Theorem (C.).**

*Let $\mathcal{C}$ be a cocomplete category with pullbacks and a terminal object and such that pullbacks preserve colimits. Let then $\tau$ be a subcanonical Grothendieck topology on $\mathcal{C}$.*
*Then the quotient prestack $[X/G]$ is a stack.*

# The main theorem

**Theorem (C.).**

*Let $\mathcal{C}$ be a cocomplete category with pullbacks and a terminal object and such that pullbacks preserve colimits. Let then $\tau$ be a subcanonical Grothendieck topology on $\mathcal{C}$.*
*Then the quotient prestack $[X/G]$ is a stack.*

We show that $[X/G]$ is a stack when $\tau = \kappa$.
This implies that it is a stack every time $\tau$ is subcanonical, since the principal bundles with respect to a subcanonical topology are principal bundles in $(\mathcal{C}, k)$ as well.

# Gluing of morphisms

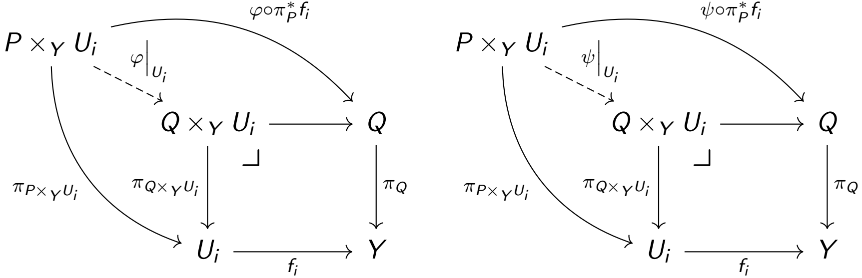
Let $(P,\alpha),(Q,\beta)\in[X/G](Y)$ and for every $i\in I$ let $\varphi_i\colon (P,\alpha)\big|_{U_i}\to(Q,\alpha)\big|_{U_i}$ be a morphism of $\mathcal{C}$ such that for every $i,j\in I$ we have $\varphi_i\big|_{U_{ij}}=\varphi_j\big|_{U_{ij}}$.

To construct $\eta\colon (P,\alpha)\to(Q,\beta)$ such that $\eta\big|_{U_i}=\varphi_i$ for every $i\in I$ we use the universal property of the following coequalizer

$$\begin{array}{ccccc}
\left(\coprod\limits_{i\in I}P\times_Y U_i\right)\times_P\left(\coprod\limits_{i\in I}P\times_Y U_i\right) & \rightrightarrows & \coprod\limits_{i\in I}P\times_Y U_i & \xrightarrow{\coprod_{i\in I}\pi_P^* f_i} & P \\
 & & & \searrow^{\coprod_{i\in I}(\pi_Q^* f_i\circ\varphi_i)} & \downarrow{\scriptstyle \eta} \\
 & & & & Q.
\end{array}$$

# Uniqueness of gluings

Let $(P,\alpha),(Q,\beta) \in [X/G](Y)$ and let $\varphi,\psi\colon (P,\alpha) \to (Q,\beta)$ be morphisms such that for every $i \in I$ $\varphi\big|_{U_i} = \psi\big|_{U_i}$.




We have $\varphi \circ \pi_P^* f_i = \psi \circ \pi_P^* f_i$ and this implies $\varphi = \psi$ because the morphisms $\{\pi_P^* f_i\}_{i\in I}$ are jointly epimorphic.

# Every descent datum is effective

Given a descent datum on the sieve sieve $S$ on $Y$, we need to define an object $(W, \alpha) \in [X/G](Y)$ and, for every morphism $Z \xrightarrow{f} Y \in S$, an isomorphism $\psi^f \colon f^*((W, \alpha)) \xrightarrow{\simeq} (W_f, \alpha_f)$ such that, given morphisms $Z' \xrightarrow{g} Z \xrightarrow{f} Y$ with $f \in S$, we have $\varphi^{f,g} \circ g^*\psi^f = \psi^{f \circ g} \circ (\varepsilon_{f,g})_{(W,\alpha)}$.

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We construct a functor $\Lambda : S \to [X/G](Y)$ that sends:

- an object $Z \xrightarrow{f} Y \in S$ to $(W_f \xrightarrow{f \circ \pi_{W_f}} Y, \alpha_f) \in [X/G](Y)$;
- a morphism $k$ from $Z \xrightarrow{f} Y$ to $P \xrightarrow{t} Y$ to the composite morphism

$$W_f \xrightarrow{(\varphi^{k,t})^{-1}} k^* W_t \xrightarrow{\pi^*_{W_t} k} W_t.$$

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$$W_f \xrightarrow{(\varphi^{k,t})^{-1}} k^* W_t \xrightarrow{\pi^*_{W_t} k} W_t.$$

We define $W := \operatorname{colim} \Lambda$ and we induce the morphism $\alpha \colon W \to X$ using the cocone given by the morphisms $\alpha_f \colon Z \to Y$ for every $f \in S$.

# Every descent datum is effective

We induce the isomorphism $\psi^f\colon f^*W \xrightarrow{\simeq} W_f$ using the universal property of the colimit $\operatorname{colim}(f^*\circ\Lambda) = f^*W$, with cocone given by

$$f^*W_t \xrightarrow{\theta_t} (f^*t)^*(W_f) \xrightarrow{\pi^*_{W_f}(f^*t)} W_f,$$

where $\theta_t$ is the composite

$$f^*W_t \xrightarrow{\simeq} (t^*f)^*(W_t) \xrightarrow{\varphi^{t,t^*f}} W_{t\circ t^*f} \xrightarrow{(\varphi^{f,f^*t})^{-1}} (f^*t)^*(W_f).$$

To prove that this is a cocone, we show that there exists a natural transformation $\Theta\colon \Lambda\circ f^* \Rightarrow (\pi_{W_f})^*\circ f^*\circ \operatorname{dom}$ of components $\theta_t$ for every $t\in S$ .