

Generalized principal bundles and quotient stacks

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PSSL 107

01/04/2023

Definition.

Let X be a topological space and let G be a topological group. A **principal G -bundle** over X is a topological space P equipped with an action $p: G \times P \rightarrow P$ and a G -equivariant continuous map $\pi_P: P \rightarrow X$, that is locally trivial, i.e. there exists an open covering $\{U_i\}_{i \in I}$ of X such that for every $i \in I$ the restriction $P|_{U_i}$ is isomorphic to $G \times U_i$ via a G -equivariant isomorphism.

Setting and preliminary definitions

Let \mathcal{C} be a category with pullbacks and terminal object T and let τ be a Grothendieck topology on it.

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Definition.

Let G be an internal group in \mathcal{C} and let X be an object of \mathcal{C} . An **action of G on X** is a morphism $x: G \times X \rightarrow X$ such that the following diagrams are commutative

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\text{id}_G \times x} & G \times X \\ m \times \text{id}_X \downarrow & & \downarrow x \\ G \times X & \xrightarrow{x} & X \end{array} \qquad \begin{array}{ccc} T \times X & \xrightarrow{e \times \text{id}_X} & G \times X \\ \text{pr}_2 \downarrow & & \downarrow x \\ X & \xrightarrow{\text{id}_X} & X, \end{array}$$

where $m: G \times G \rightarrow G$ is the internal multiplication of G and $e: T \rightarrow G$ is the internal neutral element of G .

Setting and preliminary definitions

Definition.

Let G be a group object in \mathcal{C} that acts on the objects X and Y of \mathcal{C} with actions $x: G \times X \rightarrow X$ and $y: G \times Y \rightarrow Y$ respectively. A **G -equivariant morphism** $f: X \rightarrow Y$ is a morphism in \mathcal{C} such that the following square is commutative:

$$\begin{array}{ccc} G \times X & \xrightarrow{x} & X \\ \text{id}_G \times f \downarrow & & \downarrow f \\ G \times Y & \xrightarrow{y} & Y. \end{array}$$

Actions on pullbacks

Let G be a group object of \mathcal{C} that acts on $P, Y, Z \in \mathcal{C}$ and let $f: Z \rightarrow Y$ and $g: P \rightarrow Y$ be G -equivariant morphisms.

We define an action of G on the pullback $P \times_Y Z$ using the morphism $\psi: G \times (P \times_Y Z) \rightarrow P \times_Y Z$ induced by the universal property of the pullback $P \times_Y Z$ as in the following diagram

$$\begin{array}{ccccc}
 G \times (P \times_Y Z) & \xrightarrow{\text{id}_G \times g^* f} & G \times P & & \\
 \downarrow \psi & & \searrow p & & \\
 & & P & & \\
 \downarrow f^* g & & \downarrow g & & \\
 G \times Z & \xrightarrow{z} & Z & \xrightarrow{f} & Y
 \end{array}$$

The diagram illustrates the construction of an action ψ on the pullback $P \times_Y Z$. The objects are arranged in a grid. The top row consists of $G \times (P \times_Y Z)$, $G \times P$, and P . The bottom row consists of $G \times Z$, Z , and Y . The rightmost column consists of P , Z , and Y . Morphisms are as follows:

- $\text{id}_G \times g^* f: G \times (P \times_Y Z) \rightarrow G \times P$ (top horizontal)
- $p: G \times P \rightarrow P$ (top right diagonal)
- $g^* f: P \times_Y Z \rightarrow P$ (middle horizontal)
- $\psi: G \times (P \times_Y Z) \rightarrow P \times_Y Z$ (dashed diagonal)
- $f^* g: P \times_Y Z \rightarrow G \times Z$ (middle left vertical)
- $g: P \rightarrow Y$ (right vertical)
- $z: G \times Z \rightarrow Z$ (bottom left diagonal)
- $f: Z \rightarrow Y$ (bottom horizontal)

 A right-angle symbol \perp is placed between the morphisms $g^* f$ and $f^* g$, indicating that ψ is the unique morphism induced by the universal property of the pullback $P \times_Y Z$ from the pair $(\text{id}_G \times g^* f, f^* g)$.

Locally trivial morphisms

Definition (C.).

Let $g: Y \rightarrow X$ be a morphism of \mathcal{C} . We say that g is **locally trivial** if there exists a covering $\{f_i: U_i \rightarrow X\}_{i \in I}$ of X such that for every $i \in I$ the pullback

$$\begin{array}{ccc} Y \times_X U_i & \xrightarrow{g^* f_i} & Y \\ \downarrow f_i^* g & \lrcorner & \downarrow g \\ U_i & \xrightarrow{f_i} & X \end{array}$$

is isomorphic to $G \times U_i$ via a G -equivariant isomorphism.

Principal G -bundles

Definition (C.).

Let G be an internal group in the site (\mathcal{C}, τ) and let X be an object of \mathcal{C} . A **principal G -bundle over X** is an object $P \in \mathcal{C}$ equipped with an action $p: G \times P \rightarrow P$ and a G -equivariant locally trivial morphism $\pi_P: P \rightarrow X$. Let $\pi_P: P \rightarrow X$ and $\pi_Q: Q \rightarrow X$ be principal G -bundles over X in \mathcal{C} . A **morphism of principal G -bundles over X** from $\pi_P: P \rightarrow X$ to $\pi_Q: Q \rightarrow X$ is a G -equivariant morphism $\varphi: P \rightarrow Q$ in \mathcal{C} such that

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ \pi_P \searrow & & \swarrow \pi_Q \\ & X. & \end{array}$$

Proposition (C.).

Let $\pi_P: P \rightarrow Y$ be a principal G -bundle over Y and let $f: Z \rightarrow Y$ be a G -equivariant morphism in \mathcal{C} . Then the morphism $f^\pi_P: P \times_Y Z \rightarrow Z$ is a principal G -bundle over Z .*

Closure under pullbacks

Proposition (C.).

Let $\pi_P: P \rightarrow Y$ be a principal G -bundle over Y and let $f: Z \rightarrow Y$ be a G -equivariant morphism in \mathcal{C} . Then the morphism $f^*\pi_P: P \times_Y Z \rightarrow Z$ is a principal G -bundle over Z .

The morphism $f^*\pi_P$ is G -equivariant since the square

$$\begin{array}{ccc} G \times (P \times_Y Z) & \xrightarrow{\psi} & P \times_Y Z \\ \text{id}_G \times f^*\pi_P \downarrow & & \downarrow f^*\pi_P \\ G \times P & \xrightarrow{p} & P, \end{array}$$

is commutative by definition of ψ .

Closure under pullbacks

We consider the covering $\mathcal{V} = \{g_i: V_i \rightarrow Y\}_{i \in I}$ such that for every $i \in I$ the pullback $P \times_Y V_i$ is isomorphic to $G \times V_i$ via the G -equivariant isomorphism α_i and we use it to define the covering family

$$\mathcal{U} = \{f^*g_i: V_i \times_Y Z \rightarrow Z\}_{i \in I}.$$

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$$\mathcal{U} = \{f^*g_i: V_i \times_Y Z \rightarrow Z\}_{i \in I}.$$

We show that $G \times (V_i \times_Y Z)$ satisfies the universal property of the pullback $(P \times_Y Z) \times_Z (V_i \times_Y Z)$.

$$\begin{array}{ccc} G \times (V_i \times_Y Z) & \xrightarrow{h_i} & P \times_Y Z \\ \text{pr}_2 \downarrow & & \downarrow f^* \pi_P \\ V_i \times_Y Z & \xrightarrow{f^* g_i} & Z \end{array}$$

Closure under pullbacks

For every $i \in I$, we define the morphism h_i as follows

$$\begin{array}{ccccc}
 & \text{id}_G \times g_i^* f & \rightarrow & G \times V_i & \xrightarrow{\alpha_i^{-1}} & P \times_Y V_i & \xrightarrow{\pi_P^* g_i} & P \\
 G \times (V_i \times_Y Z) & \searrow^{h_i} & & & & & & \\
 & \text{id}_G \times f^* g_i & \rightarrow & G \times Z & \xrightarrow{\text{pr}_2} & Z & \xrightarrow{f} & Y \\
 & & & & & \uparrow f^* \pi_P & & \uparrow \pi_P \\
 & & & & & P \times_Y Z & \xrightarrow{\pi_P^* f} & P
 \end{array}$$

\lrcorner

Generalized quotient prestacks

Definition (C.).

Let X be an object of \mathcal{C} and let G be a group object of \mathcal{C} that acts on X with action $x: G \times X \rightarrow X$.

The **quotient prestack** $[X/G]: \mathcal{C}^{\text{op}} \rightarrow \mathcal{Cat}$ is defined as follows:

- for every object $Y \in \mathcal{C}$ we define $[X/G](Y)$ as the category that has
 - as objects the pairs (P, α) where $\pi_P: P \rightarrow Y$ is a principal G -bundle over Y and $\alpha: P \rightarrow X$ is a G -equivariant morphism;
 - as morphisms from (P, α) to (Q, β) the morphisms of principal G -bundles $\varphi: P \rightarrow Q$ such that

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ \alpha \searrow & & \swarrow \beta \\ & X & \end{array}$$

Definition (C.).

- for every morphism $f: Z \rightarrow Y$ in \mathcal{C} , we define

$$[X/G](f): [X/G](Y) \rightarrow [X/G](Z)$$

as the functor that sends

- an object $(P, \alpha) \in [X/G](Y)$ to the pair $(P \times_Y Z, \alpha \circ \pi_P^* f)$, where $P \times_Y Z$ is the pullback of f and π_P ;
- a morphism $\varphi: (P, \alpha) \rightarrow (Q, \beta)$ to the morphism $[X/G](f)(\varphi)$ defined as follows

The diagram shows the relationship between various objects and morphisms in the quotient stack construction. It consists of the following components:

- Top-left node:** $P \times_Y Z$
- Top-right node:** Q
- Middle-right node:** $Q \times_Y Z$
- Bottom-left node:** Z
- Bottom-right node:** Y

The morphisms and their labels are:

- A curved arrow from $P \times_Y Z$ to Q labeled $\varphi \circ \pi_P^* f$.
- A curved arrow from $P \times_Y Z$ to Z labeled $f^* \pi_P$.
- A dashed arrow from $P \times_Y Z$ to $Q \times_Y Z$ labeled $[X/G](f)(\varphi)$.
- A horizontal arrow from $Q \times_Y Z$ to Q labeled $\pi_Q^* f$.
- A vertical arrow from $Q \times_Y Z$ to Z labeled $f^* \pi_Q$.
- A vertical arrow from Q to Y labeled π_Q .
- A horizontal arrow from Z to Y labeled f .
- A right-angle symbol (\perp) is placed between the vertical arrows $f^* \pi_Q$ and π_Q , indicating that $Q \times_Y Z$ is the pullback of f and π_Q .

Well-definedness of $[X/G]$

Proposition (C.).

The quotient prestack $[X/G]: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}at$ is well-defined.

Given $f: Z \rightarrow Y$ the assignment

$$[X/G](f): [X/G](Y) \rightarrow [X/G](Z)$$

- is well-defined on objects because principal G -bundles over Y are closed under pullbacks;
- preserves identities and compositions by the universal property of the pullback.

Pseudofunctoriality of $[X/G]$

Proposition (C.).

The quotient prestack $[X/G]: \mathcal{C}^{\text{op}} \rightarrow \mathcal{Cat}$ is a prestack.

$$\iota_Y: \text{id}_{[X/G](Y)} \Rightarrow [X/G](\text{id}_Y) \quad \varepsilon_{f,g}: [X/G](g) \circ [X/G](f) \Rightarrow [X/G](f \circ g)$$

The left diagram illustrates the naturality of ι_Y . It shows a commutative square with P at the top-left and Y at the bottom-left. A curved arrow labeled id_P goes from P to P . A curved arrow labeled π_P goes from P to Y . A dashed arrow labeled $\iota(P, \alpha)$ goes from P to $P \times_Y Y$. A solid arrow goes from $P \times_Y Y$ to P . A solid arrow goes from $P \times_Y Y$ to Y . A solid arrow labeled id_Y goes from Y to Y . A solid arrow labeled π_P goes from P to Y . A solid arrow labeled \lrcorner goes from $P \times_Y Y$ to Y .

The right diagram illustrates the naturality of $\varepsilon_{f,g}$. It shows a commutative square with $P \times_Y W$ at the top-left and W at the bottom-left. A curved arrow labeled $\pi_P^*(g \circ f)$ goes from $P \times_Y W$ to P . A curved arrow labeled $(g \circ f)^* \pi_P$ goes from $P \times_Y W$ to W . A dashed arrow labeled $((\varepsilon_{f,g})_{(P, \alpha)})^{-1}$ goes from $P \times_Y W$ to $(P \times_Y Z) \times_Z W$. A solid arrow goes from $(P \times_Y Z) \times_Z W$ to $P \times_Y Z$. A solid arrow goes from $(P \times_Y Z) \times_Z W$ to W . A solid arrow goes from $P \times_Y Z$ to P . A solid arrow goes from $P \times_Y Z$ to Z . A solid arrow goes from W to Z labeled g . A solid arrow goes from Z to Y labeled f . A solid arrow goes from P to Y labeled π_P . A solid arrow labeled \lrcorner goes from $P \times_Y Z$ to Z . A solid arrow labeled \lrcorner goes from $(P \times_Y Z) \times_Z W$ to W .

Remark.

The quotient prestack $[X/G]$ doesn't necessarily take values in $\mathcal{G}pd$.

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Definition (C.).

Let \mathcal{C} be a category with terminal object T and let G be a group object in \mathcal{C} . We call **classifying prestack** the prestack $[T/G]$ and we denote it BG .

Definition.

Let $\mathcal{F}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}at$ be a prestack and let S be a sieve on $Y \in \mathcal{C}$.

A **descent datum on S for \mathcal{F}** is an assignment for every morphism $Z \xrightarrow{f} Y$ in S of an object $W_f \in \mathcal{F}(Z)$ and, for every pair of composable morphisms $Z' \xrightarrow{g} Z \xrightarrow{f} Y$ with $f \in S$, of an isomorphism $\varphi^{f,g}: g^* W_f \xrightarrow{\simeq} W_{g \circ f}$ such that, given morphisms $Z'' \xrightarrow{h} Z' \xrightarrow{g} Z \xrightarrow{f} Y$ with $f \in S$, the following diagram is commutative

$$\begin{array}{ccc}
 h^*(g^* W_f) & \xrightarrow{h^* \varphi^{f,g}} & h^*(W_{f \circ g}) \\
 \downarrow \wr & & \downarrow \varphi^{f \circ g, h} \\
 (g \circ h)^*(W_f) & \xrightarrow{\varphi^{f, g \circ h}} & W_{f \circ g \circ h}
 \end{array}$$

Definition.

This descent datum is called **effective** if there exist an object $W \in \mathcal{F}(Y)$ and, for every morphism $Z \xrightarrow{f} Y \in S$, an isomorphism

$$\psi^f: f^*(W) \xrightarrow{\simeq} W_f$$

such that, given morphisms $Z' \xrightarrow{g} Z \xrightarrow{f} Y$ with $f \in S$, the following diagram is commutative

$$\begin{array}{ccc} g^*(f^*(W)) & \xrightarrow{g^*\psi^f} & g^*(W_f) \\ \downarrow \wr & & \downarrow \varphi^{f,g} \\ (f \circ g)^*W & \xrightarrow{\psi^{f \circ g}} & W_{f \circ g}. \end{array}$$

Definition of stack

Definition.

A prestack $\mathcal{F}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}at$ is a **stack** if it satisfies the following conditions:

- Every descent datum for \mathcal{F} is effective;
- (*Gluing of morphisms*) Given a covering family $\mathcal{U} = \{f_i: U_i \rightarrow U\}_{i \in I}$, objects x and y of $\mathcal{F}(U)$ and morphisms $\varphi_i: x|_{U_i} \rightarrow y|_{U_i}$ in $\mathcal{F}(U_i)$ for every $i \in I$ such that for every $i, j \in I$ $\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$, there exists a morphism $\eta: x \rightarrow y$ such that $\eta|_{U_i} = \varphi_i$;
- (*Uniqueness of gluings*) Given a covering family $\mathcal{U} = \{f_i: U_i \rightarrow U\}_{i \in I}$, objects x and y of $\mathcal{F}(U)$ and morphisms $\varphi, \psi: x \rightarrow y$ such that for every $i \in I$ $\varphi|_{U_i} = \psi|_{U_i}$, then $\varphi = \psi$.

Canonical topology on a site

Definition.

Let \mathcal{C} be a category. The **canonical topology** κ on the category \mathcal{C} is the finest Grothendieck topology on \mathcal{C} such that every representable presheaf of \mathcal{C} is a sheaf on the site (\mathcal{C}, κ) .

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Definition.

Let \mathcal{C} be a category and let X be an object of \mathcal{C} . A sieve S on X is a **colim sieve** if $X = \operatorname{colim}_S \operatorname{dom}$, where $\operatorname{dom}: S \rightarrow \mathcal{C}$ is the domain functor. Moreover, S is a **universal colim sieve** if for every morphism $f: Y \rightarrow X$ in \mathcal{C} the sieve f^*S on Y is a colim sieve.

Proposition (Johnstone, Lester).

The sieves for the canonical topology are exactly the universal colim sieves.

Basis for the canonical topology

Definition.

A morphism $f: C \rightarrow D$ in \mathcal{C} is called an **effective epimorphism** if it is the coequalizer of its kernel pair. Moreover, it is called a **universal effective epimorphism** if its base change along any morphism is an effective epimorphism.

Theorem (Lester).

Let \mathcal{C} be a cocomplete category with pullbacks and a terminal object and such that pullbacks preserve colimits. Then the family $\{f_i: Y_i \rightarrow X\}_{i \in I}$ is a covering family for X in κ if and only if the morphism

$$\coprod_{i \in I} f_i: \coprod_{i \in I} Y_i \rightarrow X$$

is a universal effective epimorphism.

The main theorem

Theorem (C.).

Let \mathcal{C} be a cocomplete category with pullbacks and a terminal object and such that pullbacks preserve colimits. Let then τ be a subcanonical Grothendieck topology on \mathcal{C} .

Then the quotient prestack $[X/G]$ is a stack.

The main theorem

Theorem (C.).

Let \mathcal{C} be a cocomplete category with pullbacks and a terminal object and such that pullbacks preserve colimits. Let then τ be a subcanonical Grothendieck topology on \mathcal{C} .

Then the quotient prestack $[X/G]$ is a stack.

We show that $[X/G]$ is a stack when $\tau = \kappa$.

This implies that it is a stack every time τ is subcanonical, since the principal bundles with respect to a subcanonical topology are principal bundles in (\mathcal{C}, k) as well.

Gluing of morphisms

Let $(P, \alpha), (Q, \beta) \in [X/G](Y)$ and for every $i \in I$ let

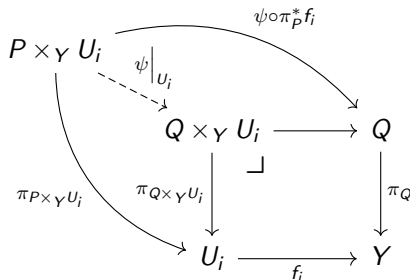
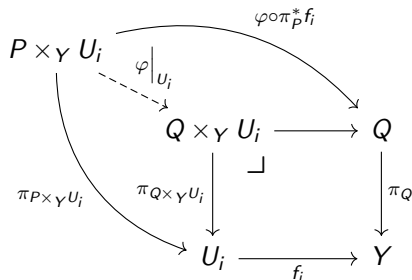
$\varphi_i: (P, \alpha)|_{U_i} \rightarrow (Q, \alpha)|_{U_i}$ be a morphism of \mathcal{C} such that for every $i, j \in I$ we have $\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$.

To construct $\eta: (P, \alpha) \rightarrow (Q, \beta)$ such that $\eta|_{U_i} = \varphi_i$ for every $i \in I$ we use the universal property of the following coequalizer

$$\begin{array}{ccc}
 (\coprod_{i \in I} P \times_Y U_i) \times_P (\coprod_{i \in I} P \times_Y U_i) & \rightrightarrows & \coprod_{i \in I} P \times_Y U_i \xrightarrow{\coprod_{i \in I} \pi_P^* f_i} P \\
 & \searrow \coprod_{i \in I} (\pi_Q^* f_i \circ \varphi_i) & \downarrow \eta \\
 & & Q.
 \end{array}$$

Uniqueness of gluings

Let $(P, \alpha), (Q, \beta) \in [X/G](Y)$ and let $\varphi, \psi: (P, \alpha) \rightarrow (Q, \beta)$ be morphisms such that for every $i \in I$ $\varphi|_{U_i} = \psi|_{U_i}$.



We have $\varphi \circ \pi_P^* f_i = \psi \circ \pi_P^* f_i$ and this implies $\varphi = \psi$ because the morphisms $\{\pi_P^* f_i\}_{i \in I}$ are jointly epimorphic.

Every descent datum is effective

Given a descent datum on the sieve S on Y , we need to define an object $(W, \alpha) \in [X/G](Y)$ and, for every morphism $Z \xrightarrow{f} Y \in S$, an isomorphism $\psi^f: f^*((W, \alpha)) \xrightarrow{\sim} (W_f, \alpha_f)$ such that, given morphisms $Z' \xrightarrow{g} Z \xrightarrow{f} Y$ with $f \in S$, we have $\varphi^{f,g} \circ g^* \psi^f = \psi^{f \circ g} \circ (\varepsilon_{f,g})_{(W, \alpha)}$.

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We construct a functor $\Lambda : S \rightarrow [X/G](Y)$ that sends:

- an object $Z \xrightarrow{f} Y \in S$ to $(W_f \xrightarrow{f \circ \pi_{W_f}} Y, \alpha_f) \in [X/G](Y)$;
- a morphism k from $Z \xrightarrow{f} Y$ to $P \xrightarrow{t} Y$ to the composite morphism

$$W_f \xrightarrow{(\varphi^{k,t})^{-1}} k^* W_t \xrightarrow{\pi_{W_t}^* k} W_t.$$

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$$W_f \xrightarrow{(\varphi^{k,t})^{-1}} k^* W_t \xrightarrow{\pi_{W_t}^* k} W_t.$$

We define $W := \operatorname{colim} \Lambda$ and we induce the morphism $\alpha : W \rightarrow X$ using the cocone given by the morphisms $\alpha_f : Z \rightarrow Y$ for every $f \in S$.

Every descent datum is effective

We induce the isomorphism $\psi^f: f^*W \xrightarrow{\cong} W_f$ using the universal property of the colimit $\operatorname{colim}(f^* \circ \Lambda) = f^*W$, with cocone given by

$$f^*W_t \xrightarrow{\theta_t} (f^*t)^*(W_f) \xrightarrow{\pi_{W_f}^*(f^*t)} W_f,$$

where θ_t is the composite

$$f^*W_t \xrightarrow{\cong} (t^*f)^*(W_t) \xrightarrow{\varphi^{t,t^*f}} W_{t \circ t^*f} \xrightarrow{(\varphi^{f,f^*t})^{-1}} (f^*t)^*(W_f).$$

To prove that this is a cocone, we show that there exists a natural transformation $\Theta: \Lambda \circ f^* \Rightarrow (\pi_{W_f})^* \circ f^* \circ \operatorname{dom}$ of components θ_t for every $t \in S$.