Principal bundles in sites and 2-sites and quotient stacks

Elena Caviglia

University of Leicester

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Principal bundles in $\mathcal{Top}$

Definition.

Let Y be a topological space and let G be a topological group.

A **principal** G**-bundle** over Y is a topological space P equipped with an action $p\colon G \times P \to P$ and a G-equivariant continuous map $\pi_P\colon P \to Y$, that is locally trivial,

i.e. there exists an open covering $\{U_i\}_{i\in I}$ of Y such that for every $i \in I$ the restriction $P\big|_{U_i}$ is isomorphic to $G \times U_i$ via a G-equivariant isomorphism.

Quotient stacks in $\mathcal{Top}$

Let $X \in \mathcal{Top}$ and let G be a topological group that acts on it.

The **quotient stack** $[X/G]\colon \mathcal{Top}^{\mathrm{op}} \to \mathcal{Gpd}$ sends:

- every topological space Y to the groupoid $[X/G](Y)$ of pairs (P, α) where $\pi_P\colon P \to Y$ is a principal G-bundle over Y and $\alpha\colon P \to X$ is a G-equivariant continuous map;

- every continuous map $f\colon Z \to Y$ to the functor

$$[X/G](f) = f^*\colon [X/G](Y) \to [X/G](Z)$$

defined via pullback along f.

Plan of the talk

(1) Principal bundles and quotient stacks in sites

E. Caviglia.
Generalized principal bundles and quotient stacks.
Theory and Applications of Categories, 39:567–597, 2023.

(2) Principal 2-bundles and quotient 2-stacks in 2-sites

work in progress

Locally trivial morphisms

Let $\mathcal{C}$ be a finitely complete category and let τ be a Grothendieck topology on it. Let G be an internal group in $\mathcal{C}$.

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Definition (C.).

Let $g\colon Y \to X$ be a morphism of $\mathcal{C}$. We say that g is **locally trivial** if there exists a covering $\{f_i\colon U_i \to X\}_{i \in I}$ of X such that for every $i \in I$ the pullback

$$\begin{array}{ccc} Y \times_X U_i & \xrightarrow{g^* f_i} & Y \\ {\scriptstyle f_i^* g}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ U_i & \xrightarrow[f_i]{} & X \end{array}$$

is isomorphic to $G \times U_i$ via a G-equivariant isomorphism.

Principal G-bundles

Definition (C.).

A **principal G-bundle over** X is an object $P \in \mathcal{C}$ equipped with an action $p \colon G \times P \to P$ and a G-equivariant locally trivial morphism $\pi_P \colon P \to X$.

A **morphism of principal G-bundles over** X from $\pi_P \colon P \to X$ to $\pi_Q \colon Q \to X$ is a G-equivariant morphism $\varphi \colon P \to Q$ in $\mathcal{C}$ such that $\pi_Q \circ \varphi = \pi_P$.

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Proposition (C.).

Principal G-bundles are closed under pullbacks.

Generalized quotient prestacks

Definition (C.).

Let X be an object of $\mathcal{C}$ and let G be a group object of $\mathcal{C}$ that acts on X with action $x\colon G \times X \to X$.

The **quotient prestack** $[X/G]\colon \mathcal{C}^{\text{op}} \to \mathcal{C}at$ is defined as follows:

- for every object $Y \in \mathcal{C}$ we define $[X/G](Y)$ as the category of pairs (P, α) where $\pi_P\colon P \to Y$ is a principal G-bundle over Y and $\alpha\colon P \to X$ is a G-equivariant morphism;
- for every morphism $f\colon Z \to Y$ in $\mathcal{C}$, we define the functor

$$[X/G](f) = f^*\colon [X/G](Y) \to [X/G](Z)$$

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Proposition (C.).

The quotient prestack $[X/G]\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ *is a prestack.*

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The quotient prestack $[X/G]$ doesn't necessarily take values in $\mathcal{G}pd$.

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Remark.

The quotient prestack $[X/G]$ doesn't necessarily take values in $\mathcal{G}pd$.

Definition (C.).

Let $\mathcal{C}$ be a category with terminal object T and let G be a group object in $\mathcal{C}$. We call **classifying prestack** the prestack $[T/G]$ and we denote it $\mathcal{B}G$.

$[X/G]$ is a stack

Theorem (C.).

Let $\mathcal{C}$ be a cocomplete and finitely complete category such that pullbacks preserve colimits. Let then τ be a subcanonical Grothendieck topology on $\mathcal{C}$.

Then the quotient prestack $[X/G]$ is a stack.

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Then the quotient prestack $[X/G]$ is a stack.

We show that $[X/G]$ is a stack when τ is the canonical topology.

This implies that it is a stack every time τ is subcanonical.

A key idea of the proof

Definition.

Let $\mathcal{C}$ be a category and let X be an object of $\mathcal{C}$. A sieve S on X is a **colim sieve** if $X = \operatorname{colim}_S \operatorname{dom}$, where $\operatorname{dom}\colon S \to \mathcal{C}$ is the domain functor.

Moreover, S is a **universal colim sieve** if for every morphism $f\colon Y \to X$ in $\mathcal{C}$ the sieve f^*S on Y is a colim sieve.

Proposition (Johnstone, Lester).

The sieves for the canonical topology are exactly the universal colim sieves.

Dimension 2: setting

We will consider a 2-category $\mathcal{K}$ that has all finite flexible limits.

Theorem (Power).

Every bicategory with finite bilimits is biequivalent to a 2-category with finite flexible limits.

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We want to put a Grothendieck topology on $\mathcal{K}$.

Definition (Street).

A **bisieve** over $\mathcal{X} \in \mathcal{K}$ is a fully faithful arrow $R \to \mathcal{K}(-, \mathcal{X})$ in $[\mathcal{K}^{\mathrm{op}}, \mathcal{C}at]$.

Grothendieck topology on a 2-category

Definition (Street).

A topology τ on $\mathcal{K}$ is an assignment for each object $\mathcal{Y} \in \mathcal{K}$ of a set $\tau(\mathcal{Y})$ of bisieves satisfying the following conditions:

(T0) the identity of $\mathcal{K}(-, \mathcal{Y})$ is in $\tau(\mathcal{Y})$;

(T1) for all $\alpha \colon R \to \mathcal{K}(-, \mathcal{Y})$ in $\tau(\mathcal{Y})$ and all arrows $f \colon \mathcal{Z} \to \mathcal{Y}$ in $\mathcal{K}$, the bi-iso-comma object

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{K}(-, \mathcal{Z}) \\ \big\downarrow & \overset{\simeq}{\swarrow} & \big\downarrow{\scriptstyle -\circ f} \\ R & \underset{\alpha}{\longrightarrow} & \mathcal{K}(-, \mathcal{Y}) \end{array}$$

has the top arrow is in $\tau(\mathcal{Z})$;

(T2) being a bisieve in τ can be checked locally.

Actions of 2-groups

Definition.

Let $\mathcal{G}$ be a (coherent) 2-group in $\mathcal{K}$ and let $\mathcal{X}$ be an object of $\mathcal{K}$. An action of $\mathcal{G}$ on $\mathcal{X}$ is a morphism $x\colon \mathcal{G}\times\mathcal{X}\to\mathcal{X}$ together with invertible 2-cells

$$
\begin{array}{ccc}
\mathcal{G}\times\mathcal{G}\times\mathcal{X} & \xrightarrow{\ \mathrm{id}_{\mathcal{G}}\times x\ } & \mathcal{G}\times\mathcal{X} \\
{\scriptstyle m\times\mathrm{id}_{\mathcal{X}}}\downarrow & \overset{\cong}{\swarrow}\,\mu & \downarrow{\scriptstyle x} \\
\mathcal{G}\times\mathcal{X} & \xrightarrow[\ x\ ]{} & \mathcal{X}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{T}\times\mathcal{X} & \xrightarrow{\ e\times\mathrm{id}_{\mathcal{X}}\ } & \mathcal{G}\times\mathcal{X} \\
{\scriptstyle \mathrm{pr}_2}\downarrow & \overset{\cong}{\swarrow}\,\nu & \downarrow{\scriptstyle x} \\
\mathcal{X} & \xrightarrow[\ \mathrm{id}_{\mathcal{X}}\ ]{} & \mathcal{X}
\end{array}
$$

that satisfy some coherence axioms (involving the associator and the unitor 2-cells of $\mathcal{G}$).

$\mathcal{G}$-equivariant morphisms

Definition.

A morphism $f\colon \mathcal{P} \to \mathcal{Q}$ in $\mathcal{K}$ is $\mathcal{G}$-equivariant if there exists an invertible 2-cell

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{P} & \xrightarrow{\;p\;} & \mathcal{P} \\ {\scriptstyle \mathrm{id}_{\mathcal{G}} \times f}\downarrow & \overset{\cong}{\underset{\lambda}{\swarrow}} & \downarrow{\scriptstyle f} \\ \mathcal{G} \times \mathcal{Q} & \xrightarrow[\;q\;]{} & \mathcal{Q} \end{array}$$

that satisfies some coherence axioms (involving the 2-cells associated to the actions p and q).

2-locally trivial morphisms

Definition (C.).

Let $\mathcal{X}$ and $\mathcal{Y}$ be objects of $\mathcal{K}$ with fixed actions of $\mathcal{G}$ on them. We say that the morphism $g\colon \mathcal{Y} \to \mathcal{X}$ is **2-locally trivial** if there exists a bisieve $S\colon R \to \mathcal{K}(-,\mathcal{X})$ in $\tau(\mathcal{X})$ such that, for every $f\colon \mathcal{U} \to \mathcal{X}$ in the bisieve, the bi-iso-comma object

$$\begin{array}{ccc} \mathcal{Y} \times_{\mathcal{X}} \mathcal{U} & \xrightarrow{g^* f} & \mathcal{Y} \\ {\scriptstyle f^* g}\big\downarrow & \overset{\simeq}{\swarrow}{\scriptstyle \lambda} & \big\downarrow{\scriptstyle g} \\ \mathcal{U} & \xrightarrow[f]{} & \mathcal{X} \end{array}$$

is equivalent to the product $\mathcal{G} \times \mathcal{U}$ over $\mathcal{U}$ via a $\mathcal{G}$-equivariant equivalence.

Principal $\mathcal{G}$-2-bundles

Definition (C.).

Let $\mathcal{Y}$ be an object of $\mathcal{K}$. A **principal $\mathcal{G}$-2-bundle over $\mathcal{Y}$** is an object $\mathcal{P} \in \mathcal{K}$ equipped with an action $p\colon \mathcal{G} \times \mathcal{P} \to \mathcal{P}$ and a $\mathcal{G}$-equivariant 2-locally trivial morphism $\pi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{Y}$.

A **morphism of principal $\mathcal{G}$-2-bundles over $\mathcal{Y}$** from $\pi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{Y}$ to $\pi_{\mathcal{Q}}\colon \mathcal{Q} \to \mathcal{Y}$ is a $\mathcal{G}$-equivariant morphism $\varphi\colon \mathcal{P} \to \mathcal{Q}$ in $\mathcal{K}$ together with a 2-cell $\gamma_\varphi\colon \pi_{\mathcal{Q}} \circ \varphi \overset{\simeq}{\Rightarrow} \pi_{\mathcal{P}}$.

A **2-cell of principal $\mathcal{G}$-2-bundles over $\mathcal{Y}$** from $\varphi\colon \mathcal{P} \to \mathcal{Q}$ to $\psi\colon \mathcal{P} \to \mathcal{Q}$ is a $\mathcal{G}$-equivariant 2-cell $\Gamma\colon \varphi \Rightarrow \psi$ such that $\gamma_\psi \circ \Gamma = \gamma_\varphi$.

Principal $\mathcal{G}$-2-bundles

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Let $\mathcal{Y}$ be an object of $\mathcal{K}$. A **principal $\mathcal{G}$-2-bundle over $\mathcal{Y}$** is an object $\mathcal{P} \in \mathcal{K}$ equipped with an action $p\colon \mathcal{G} \times \mathcal{P} \to \mathcal{P}$ and a $\mathcal{G}$-equivariant 2-locally trivial morphism $\pi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{Y}$.

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Proposition (C.).

Principal $\mathcal{G}$-2-bundles are closed under bi-iso-comma objects.

Quotient pre-2-stacks

Definition (C.).

The **quotient pre-2-stack** $[\mathcal{X}/\mathcal{G}]\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{G}ray$ is defined as follows:

- for every object $\mathcal{Y} \in \mathcal{K}$ we define $[\mathcal{X}/\mathcal{G}](\mathcal{Y})$ as the 2-category of pairs $(\mathcal{P}, \alpha)$ where $\pi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{Y}$ is a principal $\mathcal{G}$-2-bundle over $\mathcal{Y}$ and $\alpha\colon \mathcal{P} \to \mathcal{X}$ is a $\mathcal{G}$-equivariant morphism;
- for every morphism $f\colon \mathcal{Z} \to \mathcal{Y}$ in $\mathcal{K}$, we define the 2-functor
$$[\mathcal{X}/\mathcal{G}](f) = f^*\colon [\mathcal{X}/\mathcal{G}](\mathcal{Y}) \to [\mathcal{X}/\mathcal{G}](\mathcal{Z})$$
via iso-comma object along f;
- for every 2-cell $\Lambda\colon f \Rightarrow g\colon \mathcal{Z} \to \mathcal{Y}$, we define $[\mathcal{X}/\mathcal{G}](\Lambda)$ using the universal property of the iso-comma object.

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Proposition (C.).

The quotient pre-2-stack $[\mathcal{X}/\mathcal{G}]\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{G}ray$ *is a 3-pseudofunctor.*

Definition (C.).

Let $(\mathcal{K}, \tau)$ be a bisite. A trihomomorphism $F : \mathcal{K}^{\text{op}} \to \mathcal{B}icat$ is a **2-stack** if for every object $\mathcal{C} \in \mathcal{K}$ and every bisieve $S \colon R \Rightarrow \mathcal{K}(-, \mathcal{C})$ in $\tau(\mathcal{C})$ the pseudofunctor

$$- \circ S \colon \mathcal{T}ricat(\mathcal{K}^{\text{op}}, \mathcal{B}icat)(\mathcal{K}(-, \mathcal{C}), F) \longrightarrow \mathcal{T}ricat(\mathcal{K}^{\text{op}}, \mathcal{B}icat)(S, F)$$

is a biequivalence.

This is equivalent (AC) to:

(1) surjective on equivalence classes of objects;

(2) essentially surjective on morphisms;

(3) fully-faithful on 2-cells.

$[\mathcal{X}/\mathcal{G}]$ is a 2-stack

Theorem (C., work in progress).

If τ is subcanonical, then $[\mathcal{X}/\mathcal{G}]$ is a 2-stack.

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Key idea of the proof:

Proposition (C.).

If τ is a subcanonical topology on $\mathcal{K}$ and S is a bisieve in $\tau(\mathcal{Y})$, then S is a **bicolim bisieve**, *i.e.* $\mathcal{Y} = \text{bicolim}_S\,\text{dom}$.