Principal bundles in sites and 2-sites and quotient stacks

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Abstract

Principal bundles over topological spaces are an important and useful notion in geometry, with links to cohomology theories. We introduce a notion of principal bundle that makes sense in any site that has all pullbacks and a terminal object. Using these generalized principal bundles we explicitly construct quotient prestacks. Moreover, we show that, if the site satisfies some mild conditions, these generalized quotient prestacks are stacks. In order to generalize this theory to dimension two, we introduce a notion of principal 2-bundle that makes sense in a 2-site. We upgrade group objects to 2-groups and pullbacks to bi-iso-comma objects. In analogy to the one-dimensional case, we use principal 2-bundles to construct quotient pre-2-stacks.

Dimension one

Let $\mathcal{C}$ be a category with pullbacks and terminal object $*$ and let τ be a Grothendieck topology on it.

Generalized principal bundles

Let G be a group object in $\mathcal{C}$.

Definition. Let X and Y be objects of $\mathcal{C}$ with fixed actions of G on them. We say that the morphism $g\colon Y \to X$ is **locally trivial** if there exists a covering $\{f_i\colon U_i \to X\}_{i\in I}$ of X such that for every $i \in I$ the pullback

$$\begin{array}{ccc} Y\times_X U_i & \xrightarrow{g^*f_i} & Y \\ {\scriptstyle f_i^*g}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle g} \\ U_i & \xrightarrow[f_i]{} & X \end{array}$$

is isomorphic to the product $G \times U_i$ over U_i via a G-equivariant isomorphism.

Definition. Let X be an object of $\mathcal{C}$. A **principal G-bundle over X** is an object $P \in \mathcal{C}$ equipped with an action $p\colon G \times P \to P$ and a G-equivariant locally trivial morphism $\pi_P\colon P \to X$.

A **morphism of principal G-bundles over X** from $\pi_P\colon P \to X$ to $\pi_Q\colon Q \to X$ is a G-equivariant morphism $\varphi\colon P \to Q$ in $\mathcal{C}$ such that $\pi_Q \circ \varphi = \pi_P$.

Remark. This notion of principal bundle become the usual one in both the topological and algebraic case.

Proposition. *Principal G-bundles are closed under pullbacks.*

Generalized quotient prestacks

Quotient prestacks are commonly thought as stackifications of presheaves of action groupoids, but an explicit construction that uses presheaves of groupoids of principal bundles equipped with equivariant maps to a fixed space is also present in the literature in the algebraic case and in the differentiable case. We perform the same construction in the internal context to define quotient prestacks.

Definition. The **quotient prestack** $[X/G]\colon \mathcal{C}^{\text{op}} \to \mathcal{C}at$ is defined as follows:

- for every object $Y \in \mathcal{C}$ we define $[X/G](Y)$ as the category that has
  - as objects the pairs (P, α) where $\pi_P\colon P \to Y$ is a principal G-bundle over Y and $\alpha\colon P \to X$ is a G-equivariant morphism;
  - as morphisms from (P, α) to (Q, β) the morphisms of principal G-bundles $\varphi\colon P \to Q$ such that $\beta \circ \varphi = \alpha$.
- for every morphism $f\colon Z \to Y$ in $\mathcal{C}$, we define $[X/G](f)\colon [X/G](Y) \to [X/G](Z)$ as the change of base functor that sends
  - an object $(P, \alpha) \in [X/G](Y)$ to the pair $(P \times_Y Z, \alpha \circ \pi_P^* f)$, where $P \times_Y Z$ is the pullback of f and π_P;
  - a morphism $\varphi\colon (P, \alpha) \to (Q, \beta)$ to the morphism $[X/G](f)(\varphi)$ induced by the universal property of $P \times_Y Z$ considering the pair $(\varphi \circ \pi_P^* f, f^*\pi_P)$.

The quotient prestack $[*/G]$ is called **classifying prestack**.

Proposition. *The quotient prestack $[X/G]\colon \mathcal{C}^{\text{op}} \to \mathcal{C}at$ is a pseudofunctor.*

Remark. The quotient prestack $[X/G]$ does not necessarily take values in $\mathcal{G}pd$. This is due to the fact that in a generic site it is not always possible to define morphisms locally.

Generalized quotient stacks

> **Theorem** ([1]). *Let $\mathcal{C}$ be a cocomplete category with pullbacks and a terminal object and such that pullbacks preserve colimits. Let then τ be a subcanonical Grothendieck topology on $\mathcal{C}$. Then the quotient prestack $[X/G]$ is a stack.*

Ideas of the proof. We show that $[X/G]$ is a stack when τ is the canonical topology. We use the fact that covering families for the canonical topology are exactly families of jointly universal effective epimorphisms and sieves are exactly universal colim sieves.

- **gluing of morphisms**:
  Let $(P, \alpha), (Q, \beta) \in [X/G](Y)$ and for every $i \in I$ let $\varphi_i\colon (P, \alpha)\big|_{U_i} \to (Q, \alpha)\big|_{U_i}$ be a morphism of $\mathcal{C}$ such that for every $i, j \in I$ we have $\varphi_i\big|_{U_{ij}} = \varphi_j\big|_{U_{ij}}$. To construct $\eta\colon (P, \alpha) \to (Q, \beta)$ such that $\eta\big|_{U_i} = \varphi_i$ for every $i \in I$ we use the universal property of the following coequalizer
  $$\Big(\coprod_{i\in I} P\times_Y U_i\Big) \times_P \Big(\coprod_{i\in I} P\times_Y U_i\Big) \rightrightarrows \coprod_{i\in I} P\times_Y U_i \xrightarrow{\coprod_{i\in I}\pi_P^* f_i} P, \qquad P \overset{\eta}{\dashrightarrow} Q, \qquad \coprod_{i\in I}(\pi_Q^* f_i \circ \varphi_i)\colon \coprod_{i\in I} P\times_Y U_i \to Q.$$
- **uniqueness of gluings**:
  Let $(P, \alpha), (Q, \beta) \in [X/G](Y)$ and let $\varphi, \psi\colon (P, \alpha) \to (Q, \beta)$ be morphisms such that for every $i \in I$ $\varphi\big|_{U_i} = \psi\big|_{U_i}$. By construction we have $\varphi \circ \pi_P^* f_i = \psi \circ \pi_P^* f_i$ and this implies $\varphi = \psi$ because the morphisms $\{\pi_P^* f_i\}_{i\in I}$ are jointly epimorphic.
- **every descent datum is effective**:
  Given a descent datum on the sieve sieve S on Y, we need to define an object $(W, \alpha) \in [X/G](Y)$ and, for every morphism $Z \xrightarrow{f} Y \in S$, an isomorphism $\psi^f\colon f^*((W, \alpha)) \xrightarrow{\simeq} (W_f, \alpha_f)$ such that, given morphisms $Z' \xrightarrow{g} Z \xrightarrow{f} Y$ with $f \in S$, we have $\varphi^{f,g} \circ g^*\psi^f = \psi^{f\circ g} \circ (\varepsilon_{f,g})_{(W,\alpha)}$. We define W as the colimit of the functor $\Lambda\colon S \to [X/G](Y)$ that extends to a functor the assignment given by the sieve S post-composing each W_f with f to obtain a principal bundle over Y.
  We then induce the morphism $\alpha\colon W \to X$ using the cocone given by the morphisms $\alpha_f\colon Z \to Y$ for every $f \in S$ and the isomorphism $\psi^f\colon f^*W \xrightarrow{\simeq} W_f$ using the universal property of the colimit $\operatorname{colim}(f^* \circ \Lambda) = f^*W$.

Dimension two

Let $\mathcal{K}$ be a 2-category with bi-iso-comma objects (sometimes called bipullbacks) and bi-terminal object $*$ and let τ be a 2-dimensional Grothendieck topology on it (in the sense of [2]).

Principal 2-bundles

Let $\mathcal{G}$ be a (coherent) 2-group in $\mathcal{K}$.

Definition. Let $\mathcal{X}$ and $\mathcal{Y}$ be objects of $\mathcal{C}$ with fixed actions of $\mathcal{G}$ on them. We say that the morphism $g\colon \mathcal{Y} \to \mathcal{X}$ is **2-locally trivial** if there exists a covering family $\{f_i\colon \mathcal{U}_i \to \mathcal{X}\}_{i\in I}$ of $\mathcal{X}$ such that for every $i \in I$ the bi-iso-comma object

$$\begin{array}{ccc} \mathcal{Y}\times_{\mathcal{X}} \mathcal{U} & \xrightarrow{g^*f_i} & \mathcal{Y} \\ {\scriptstyle f_i^*g}\big\downarrow & \overset{\lambda_i}{\Rrightarrow} & \big\downarrow{\scriptstyle g} \\ \mathcal{U}_i & \xrightarrow[f_i]{} & \mathcal{X} \end{array}$$

is equivalent to the biproduct $\mathcal{G} \times \mathcal{U}_i$ over $\mathcal{U}_i$ via a $\mathcal{G}$-equivariant equivalence.

Definition. Let $\mathcal{X}$ be an object of $\mathcal{K}$. A **principal $\mathcal{G}$-2-bundle over $\mathcal{X}$** is an object $\mathcal{P} \in \mathcal{C}$ equipped with an action $p\colon \mathcal{G} \times \mathcal{P} \to \mathcal{P}$ and a $\mathcal{G}$-equivariant locally trivial $\pi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{X}$.

A **morphism of principal $\mathcal{G}$-2-bundles over $\mathcal{X}$** from $\pi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{X}$ to $\pi_{\mathcal{Q}}\colon \mathcal{Q} \to \mathcal{X}$ is a $\mathcal{G}$-equivariant morphism $\varphi\colon \mathcal{P} \to \mathcal{Q}$ in $\mathcal{K}$ together with a 2-cell $\gamma_\varphi\colon \pi_{\mathcal{Q}} \circ \varphi \overset{\simeq}{\Rightarrow} \pi_{\mathcal{P}}$.

A **2-cell of principal $\mathcal{G}$-2-bundles over $\mathcal{X}$** from $\varphi\colon \mathcal{P} \to \mathcal{Q}$ to $\psi\colon \mathcal{P} \to \mathcal{Q}$ is a $\mathcal{G}$-equivariant 2-cell $\Gamma\colon \varphi \Rightarrow \psi$ such that $\gamma_\psi \circ \Gamma = \gamma_\varphi$.

Proposition. *Principal $\mathcal{G}$-bundles are closed under bi-iso-comma objects.*

Quotient pre-2-stacks

We generalize the construction of quotient prestacks to dimension two. We obtain trihomomorphisms (morphisms between tricategories) that take values in the tricategory $2\text{-}\mathcal{C}at_{\text{ps}}$ of 2-categories, pseudofunctors, pseudonatural transformations and modifications.

Definition. The **quotient pre-2-stack** $[\mathcal{X}/\mathcal{G}]\colon \mathcal{K}^{\text{op}} \to 2\text{-}\mathcal{C}at_{\text{ps}}$ is defined as follows:

- for every object $\mathcal{Y} \in \mathcal{K}$ we define $[\mathcal{X}/\mathcal{G}](\mathcal{Y})$ as the 2-category that has
  - as objects the pairs $(\mathcal{P}, \alpha)$ where $\pi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{Y}$ is a principal $\mathcal{G}$-2-bundle over $\mathcal{Y}$ and $\alpha\colon \mathcal{P} \to \mathcal{X}$ is a $\mathcal{G}$-equivariant morphism;
  - as morphisms from $(\mathcal{P}, \alpha_{\mathcal{P}})$ to $(\mathcal{Q}, \beta_{\mathcal{Q}})$ the morphisms of principal $\mathcal{G}$-2-bundles $\varphi\colon \mathcal{P} \to \mathcal{Q}$ such that there exists an invertible 2-cell $\beta_\varphi\colon \alpha_{\mathcal{Q}} \circ \varphi \overset{\simeq}{\Rightarrow} \alpha_{\mathcal{P}}$;
  - as 2-cells from φ to ψ (both morphisms from $(\mathcal{P}, \alpha_{\mathcal{P}})$ to $(\mathcal{Q}, \alpha_{\mathcal{Q}})$) the 2-cells of principal $\mathcal{G}$-2-bundles $\Gamma\colon \varphi \Rightarrow \psi$ such that $\beta_\psi \circ \Gamma = \beta_\varphi$;
- for every morphism $f\colon \mathcal{Z} \to \mathcal{Y}$ in $\mathcal{K}$, we define $[\mathcal{X}/\mathcal{G}](f)\colon [\mathcal{X}/\mathcal{G}](\mathcal{Y}) \to [\mathcal{X}/\mathcal{G}](\mathcal{Z})$ as the pseudofunctor that sends
  - an object $(\mathcal{P}, \alpha) \in [\mathcal{X}/\mathcal{G}](\mathcal{Y})$ to the pair $(\mathcal{P} \times_{\mathcal{Y}} \mathcal{Z}, \alpha \circ \pi_{\mathcal{P}}^* f)$, where $\mathcal{P} \times_{\mathcal{Y}} \mathcal{Z}$ is the bi-iso-comma object of f and $\pi_{\mathcal{P}}$;
  - a morphism $\varphi\colon (\mathcal{P}, \alpha) \to (\mathcal{Q}, \beta)$ to a morphism $[\mathcal{X}/\mathcal{G}](f)(\varphi)$ induced by the (one-dimensional) universal property of $\mathcal{P} \times_{\mathcal{Y}} \mathcal{Z}$ considering the invertible 2-cell $\gamma_\varphi \circ \lambda_{\mathcal{P}}$;
  - a 2-cell Γ from φ to ψ (both morphisms from $(\mathcal{P}, \alpha_{\mathcal{P}})$ to $(\mathcal{Q}, \alpha_{\mathcal{Q}})$) to the 2-cell $[\mathcal{X}/\mathcal{G}](f)(\Gamma)$ induced by the (two-dimensional) universal property of $\mathcal{P} \times_{\mathcal{Y}} \mathcal{Z}$ considering as 2-cells the natural ones obtained composing Γ with the invertible 2-cells given by the definition of $[\mathcal{X}/\mathcal{G}](f)(\varphi)$ and $[\mathcal{X}/\mathcal{G}](f)(\psi)$.
- for every 2-cell $\Lambda\colon f \Rightarrow g\colon \mathcal{Z} \to \mathcal{Y}$, we define $[\mathcal{X}/\mathcal{G}](\Lambda)$ as the pseudonatural transformation that has as component relative to $(\mathcal{P}, \alpha_{\mathcal{P}})$ a morphism $[\mathcal{X}/\mathcal{G}](\Lambda)_{(\mathcal{P},\alpha_{\mathcal{P}})}\colon [\mathcal{X}/\mathcal{G}](f)(\mathcal{P}, \alpha_{\mathcal{P}}) \to [\mathcal{X}/\mathcal{G}](g)(\mathcal{P}, \alpha_{\mathcal{P}})$ induced by the (one-dimensional) universal property of the bi-iso-comma object of $\pi_{\mathcal{P}}$ and g considering the 2-cell given by the pasting of the bi-iso-comma square of $\pi_{\mathcal{P}}$ and f with Λ.

The quotient pre-2-stack $[*/\mathcal{G}]$ is called **classifying pre-2-stack**.

Proposition. *The quotient pre-2-stack $[\mathcal{X}/\mathcal{G}]\colon \mathcal{K}^{\text{op}} \to 2\text{-}\mathcal{C}at_{\text{ps}}$ is a trihomomorphism.*

Is $[\mathcal{X}/\mathcal{G}]$ a 2-stack?

In analogy with the one-dimensional case, we would like to prove that, under some mild assumptions on the 2-site $(\mathcal{K}, \tau)$, the quotient pre-2-stack $[\mathcal{X}/\mathcal{G}]\colon \mathcal{K}^{\text{op}} \to 2\text{-}\mathcal{C}at_{\text{ps}}$ satisfies descent conditions.

Considering the definition of stacks given by Ross Street in [2], we give the following definition:

Definition. Let $(\mathcal{K}, \tau)$ be a bisite. A trihomomorphism $F\colon \mathcal{K}^{\text{op}} \to \mathcal{B}icat$ is a **2-stack** if for every object $\mathcal{C} \in \mathcal{K}$ and every bisieve $s\colon S \Rightarrow \mathcal{K}(-, \mathcal{C})$ the pseudofunctor

$$- \circ s\colon \mathcal{T}ricat(\mathcal{K}^{\text{op}}, \mathcal{B}icat)(\mathcal{K}(-, \mathcal{C}), F) \longrightarrow \mathcal{T}ricat(\mathcal{K}^{\text{op}}, \mathcal{B}icat)(S, F)$$

is a biequivalence. Assuming the axiom of choice, this means that $- \circ s$ is surjective on equivalence classes of objects, essentially surjective on morphisms and fully-faithful on 2-cells.

Objective. We aim to prove that, if the two-dimensional topology on the 2-category $\mathcal{K}$ is subcanonical and $\mathcal{K}$ satisfies some mild conditions, quotient pre-2-stacks are 2-stacks in the sense of this definition.

References

[1] E. Caviglia. Generalized principal bundles and quotient stacks. *Theory and Applications of Categories*, 39:567–597, 2023.

[2] R. Street. Characterization of bicategories of stacks. Category theory. Applications to algebra, logic and topology, Proc. int. Conf., Gummersbach 1981, Lect. Notes Math. 962, 282-291 (1982)., 1982.